Exercises

1. $y' = \frac{2x}{y+x^2y}$

2. $\begin{cases} \frac{dy}{dx} = \frac{1}{1-x^2} \\ y(0) = 2 \end{cases}$

3. $ty' + 2y = \sin t, t > 0$

4. A home buyer can afford to spend no more than \$1500/month on mortgage payments. Suppose the annual interest rate is 6%, and that interest is compounded continuously. Determine the maximum ammount that this buyer can afford to buy on a 20-year mortgage. (Answer: \$209.642)

5. Show that $\phi(t) = \frac{1}{t}$ is a solution of $y' + y^2 = 0$ for $t > 0$, but that $y = c\phi(t)$ is not a solution of this equation unless $c = 0$ or 1.

6. Suppose $y = y_1(t)$ is a solution of
$$y' + p(t)y = 0, \tag{1}$$
$y = y_2(t)$ is a solution of
$$y' + p(t)y = g(t). \tag{2}$$
Show that $y = y_1(t) + y_2(t)$ is also a solution of (2).

7. A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

   (a) Show that the volume $V(t)$ of water in the pond at time t satisfies the DE
   $$\frac{dV}{dt} = k - \alpha\pi \left(\frac{3\alpha}{\pi h}\right)^{\frac{2}{3}} V^{\frac{2}{3}}, \tag{3}$$
   where α is the coefficient of evaporation.
   (Hint: The volume for a cone can be computed by $V = \frac{1}{3}\pi r^2 \ell$, where r is the radius and ℓ is the depth.)

   (b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable? (Answer: $\frac{h}{a}\sqrt{\frac{k}{\alpha\pi}}$)

   (c) Find a condition on the coefficients that must be satisfied if the pond is not to overflow. (Answer: $\frac{k}{\alpha} \le \pi a^2$)

8. Given that $y_1(t) = \frac{1}{t}$ is a solution to the DE $t^2y'' + 3ty' + y = 0, t > 0$. Use the method of reduction of order to find another solution, so that they form a fundamental set of solutions.

9. Find the general solutions to the following DEs

   (a) $y'' - 2y' + y = e^{3t}$ (Answer: $y(t) = C_1e^t + C_2te^t + \frac{1}{4}e^{3t}$)

   (b) $y'' - 2y' + y = 25\cos(3t)$ (Answer: $y(t) = C_1e^t + C_2te^t - 8\cos(3t) - 6\sin(3t)$)

(c) $y'' + 4y = \cos t$ (Answer: $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{3} \cos t$)

(d) $y'' + 4y = \frac{1}{10} \cos(2t)$ (Answer: $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{40} t \sin(2t)$)

10. A 1-kg object is placed on a spring with spring constant $k = 4$ N/m. The object is pulled down 0.2 meters from the static position, and then set in motion with a downward velocity of 1 m/s. Find the period, amplitude and phase angle of its motion.

11. Solve the initial value problem

$$\begin{cases} y' + 2y = g(t) \\ y(0) = 0 \end{cases} \qquad \text{where } g(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$$

Hint: The solution $y(t)$ should be continuous at $t = 1$.

12. Find the inverse Laplace transform of $\frac{1}{(s-1)(s^2-4s+5)}$.