Basic Integration Examples for Review

The following pages contain a few standard/routine examples of substitution, by parts, and partial fractions.

Basic Substitution Examples

1. $\displaystyle\int x\cos(x^2)\,dx.$

   Solution: Let $u = x^2$, so $du = 2xdx$ (and $\frac{1}{2x}du = dx$).
   The integral becomes $\displaystyle\int \frac{1}{2}\cos(u)\,du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(x^2) + C.$

2. $\displaystyle\int \cos(x)e^{\sin(x)}\,dx.$

   Solution: Let $u = \sin(x)$, so $du = \cos(x)dx$ (and $\frac{1}{\cos(x)}du = dx$).
   The integral becomes $\displaystyle\int e^u\,du = e^u + C = e^{\sin(x)} + C.$

3. $\displaystyle\int x^2\sqrt{x^3+2}\,dx.$

   Solution: Let $u = x^3 + 2$, so $du = 3x^2dx$ (and $\frac{1}{3x^2}du = dx$).
   The integral becomes $\displaystyle\int \frac{1}{3}\sqrt{u}\,du = \int \frac{1}{3}u^{1/2}\,du = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(x^3+2)^{3/2} + C.$

4. $\displaystyle\int \frac{(\ln(x))^3}{x}\,dx.$

   Solution: Let $u = \ln(x)$, so $du = \frac{1}{x}dx$ (and $xdu = dx$).
   The integral becomes $\displaystyle\int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln(x))^4 + C.$

5. $\displaystyle\int \frac{x}{x^2+1}\,dx.$

   Solution: Let $u = x^2 + 1$, so $du = 2xdx$ (and $\frac{1}{2x}du = dx$).
   The integral becomes $\displaystyle\int \frac{1}{2}\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C.$

Integration by Parts

1. $\displaystyle\int x\cos(2x)\,dx$

   Solution: Let $u = x$ and $dv = \cos(2x)dx$. Then $du = dx$ and $v = \frac{1}{2}\sin(2x)$.
   The by parts formula gives $\displaystyle\frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)\,dx = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$.

2. $\displaystyle\int x^2\ln(x)\,dx$

   Solution: Let $u = \ln(x)$ and $dv = x^2dx$. Then $du = \frac{1}{x}dx$ and $v = \frac{1}{3}x^3$.
   The by parts formula gives $\displaystyle\frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^2\,dx = \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C$.

3. $\displaystyle\int x^2e^{-x}\,dx$

   Solution: Let $u = x^2$ and $dv = e^{-x}dx$. Then $du = 2xdx$ and $v = -e^{-x}$.
   The by parts formula gives $\displaystyle -x^2e^{-x} - \int -2xe^{-x}\,dx = -x^2e^{-x} + \int 2xe^{-x}\,dx$.
   Now we do by parts again with $u = 2x$ and $dv = e^{-x}dx$. Then $du = 2dx$ and $v = -e^{-x}$.
   The by parts formula gives $\displaystyle -x^2e^{-x} - 2xe^{-x} - \int -2e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$.

4. $\displaystyle\int e^x\sin(x)\,dx$

   Solution: Let $u = e^x$ and $dv = \sin(x)dx$. Then $du = e^xdx$ and $v = -\cos(x)$.
   The by parts formula gives $\displaystyle -e^x\cos(x) - \int -e^x\cos(x)\,dx = -e^x\cos(x) + \int e^x\cos(x)\,dx$.
   Now we do by parts again with $u = e^x$ and $dv = \cos(x)dx$. Then $du = e^xdx$ and $v = \sin(x)$.
   The by parts formula gives $\displaystyle -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx$.
   Thus, we have shown $\displaystyle\int e^x\sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx$, from which we can conclude that $\displaystyle 2\int e^x\sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) + C_0$.
   Therefore, $\displaystyle\int e^x\sin(x)\,dx = -\frac{1}{2}e^x\cos(x) + \frac{1}{2}e^x\sin(x) + C$.

Partial Fractions

1. $\displaystyle\int \frac{x-2}{(x+1)(x-4)}\,dx$

   Solution: Distinct linear terms decompose into the form $\frac{x-2}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$, which can be expanded to get $x-2 = A(x-4) + B(x+1) = (A+B)x + (-4A+B)$. Thus, $A+B=1$ and $-4A+B=-2$ which we can solve to get $A=\frac{3}{5}$ and $B=\frac{2}{5}$.
   (You can use the "cover-up" method to do this faster, ask me about this if you haven't seen it).
   Thus, we get $\displaystyle\int \frac{x-2}{(x+1)(x-4)}\,dx = \int \frac{3/5}{x+1} + \frac{2/5}{x-4}\,dx = \frac{3}{5}\ln|x+1| + \frac{2}{5}\ln|x-4| + C.$

2. $\displaystyle\int \frac{3}{(x+1)^2(x-2)}\,dx$

   Solution: We have a distinct and a repeated linear term which decompose into the form $\frac{3}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$, which can be expanded to get
   $3 = A(x+1)(x-2) + B(x-2) + C(x+1)^2 = (A+C)x^2 + (-A+B+2C)x + (-2A-2B+C)$.
   Thus, $A+C=0$, $-A+B+2C=0$ and $-2A-2B+C=3$ which we can solve to get $A=-\frac{1}{3}$, $B=-1$, $C=\frac{1}{3}$. (Again, there are many short-cuts you can use here, ask me if you don't know them).
   $\displaystyle\int \frac{3}{(x+1)^2(x-2)}\,dx = \int \frac{-1/3}{x+1} + \frac{-1}{(x+1)^2} + \frac{1/3}{x-2}\,dx = -\frac{1}{3}\ln|x+1| + \frac{1}{x+1} + \frac{1}{3}\ln|x-2| + C.$

3. $\displaystyle\int \frac{2x+1}{x(x^2+1)}\,dx$

   Solution: We have a distinct linear and an irreducible quadratic term which decompose into the form $\frac{2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$, which can be expanded to get $2x+1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + (C)x + (A)$. Thus, $A+B=0$, $C=2$ and $A=1$ which we can solve to get $A=1$, $B=-1$, $C=2$.
   $\displaystyle\int \frac{1}{x} + \frac{-x+2}{x^2+1}\,dx = \ln|x| - \int \frac{x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}(x) + C.$